



1.6.8.7 Division of complex numbers

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Introduction

- In this topic, we will
 - Define the division of one complex number by another
 - Derive this from multiplication and multiplicative inverses
 - Look at a geometric interpretation
 - Observe some properties





The multiplicative inverse

- Recall that given $z = \alpha + \beta j = |z| \angle \theta$, we have that

$$z^{-1} = \frac{\alpha - \beta j}{\alpha^2 + \beta^2} = \frac{z^*}{|z|^2} = \frac{1}{|z|} \angle (-\theta)$$





Complex division

- The division of a complex number by another may be written as

$$\frac{z}{w} = z \div w = z/w = zw^{-1}$$

- You can always leave this as zw^{-1} but we will calculate the entire expression explicitly





Complex conjugate approach

- We will use the final form to quickly find the formula:

$$zw^{-1} = z \frac{w^*}{|w|^2} = \frac{zw^*}{|w|^2}$$

- For example, to calculate $\frac{1+3j}{2-j}$, we have

$$\begin{aligned} \frac{1+3j}{2-j} &= \frac{(1+3j)(2+j)}{2^2+1^2} \\ &= \frac{-1+7j}{5} = -0.2+1.4j \end{aligned}$$





Polar representation approach

- Now, if $z = |z| \angle \theta$ and $w = |w| \angle \phi$,

$$\begin{aligned} zw^{-1} &= (|z| \angle \theta) (|w| \angle \phi)^{-1} \\ &= (|z| \angle \theta) \left(\frac{1}{|w|} \angle (-\phi) \right) \\ &= \frac{|z|}{|w|} \angle (\theta - \phi) \end{aligned}$$

- For example, to calculate $\frac{2 \angle \frac{\pi}{4}}{0.4 \angle \frac{3\pi}{4}}$, we have

$$\begin{aligned} \frac{2 \angle \frac{\pi}{4}}{0.4 \angle \frac{3\pi}{4}} &= \frac{2}{0.4} \angle \left(\frac{\pi}{4} - \frac{3\pi}{4} \right) \\ &= 5 \angle \left(-\frac{\pi}{2} \right) = -5j \end{aligned}$$





Complex division

- Let's check that $\frac{2\angle \frac{\pi}{4}}{0.4\angle \frac{3\pi}{4}} = -5j$

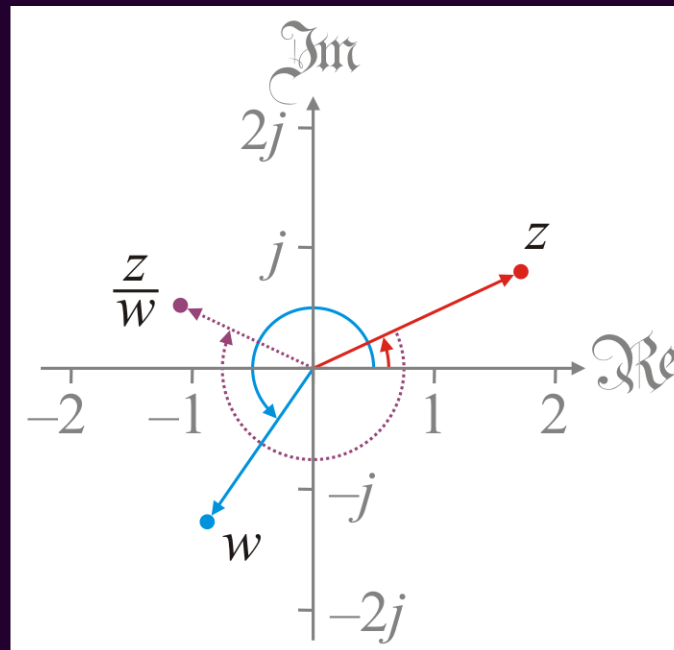
$$\begin{aligned}\frac{2\angle \frac{\pi}{4}}{0.4\angle \frac{3\pi}{4}} &= \frac{\sqrt{2} + \sqrt{2}j}{-\frac{\sqrt{2}}{5} + \frac{\sqrt{2}}{5}j} \\ &= 5 \frac{1+j}{-1+j} \\ &= 5 \frac{(1+j)(-1-j)}{2} \\ &= 5 \frac{(-1+1) + (-1-1)j}{2} = -5j\end{aligned}$$





Geometric interpretation

- The geometric interpretation of a ratio is
 - The ratio of the magnitudes
 - The angle of the dividend minus the angle of the divisor





Properties

Theorem: $\left| \frac{z}{w} \right| = \frac{|z|}{|w|}$

Proof: $\left| \frac{z}{w} \right| = |zw^{-1}|$
 $= |z| |w^{-1}|$
 $= |z| \frac{1}{|w|} = \frac{|z|}{|w|} \quad \text{QED}$





Properties

Theorem: $\left(\frac{z}{w}\right)^{-1} = \frac{w}{z}$

Proof: If $z = |z| \angle \theta$ and $w = |w| \angle \phi$, $\frac{z}{w} = \frac{|z|}{|w|} \angle (\theta - \phi)$

Thus, $\left(\frac{z}{w}\right)^{-1} = \left(\frac{|z|}{|w|}\right)^{-1} \angle -(\theta - \phi)$

$$= \frac{|w|}{|z|} \angle (\phi - \theta)$$

$$= (|w| \angle \phi) \left(\frac{1}{|z|} \angle (-\theta) \right) = \frac{w}{z} \quad \text{Q.E.D.}$$





Properties

- For example,

$$\frac{-0.9 + 1.3j}{1.1 - 0.2j} = \frac{-0.9 + 1.3j}{1.1 - 0.2j} \cdot \frac{1.1 + 0.2j}{1.1 + 0.2j}$$
$$= \frac{-1.25 + 1.25j}{1.25} = -1 + j$$

$$\frac{1.1 - 0.2j}{-0.9 + 1.3j} = \frac{1.1 - 0.2j}{-0.9 + 1.3j} \cdot \frac{-0.9 - 1.3j}{-0.9 - 1.3j}$$
$$= \frac{-1.25 - 1.25j}{2.5} = -0.5 - 0.5j$$

$$(-1 + j)(-0.5 - 0.5j) = 0.5 + 0.5 + (-0.5 + 0.5)j = 1$$





Summary

- In this topic, we've introduced division of complex numbers
 - We used two approaches
 - Complex conjugates and polar representation
 - We saw the geometric interpretation:
 - The ratio of the magnitude
 - A difference of the angles
 - We saw a number of properties





References

- [1] https://en.wikipedia.org/wiki/Complex_number#Reciprocal_and_division





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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